Surface width scaling in noise reduced Eden clusters

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The surface width scaling of Eden A clusters grown from a single aggregate site on the square lattice is investigated as a function of the noise reduction parameter. A two-exponent scaling ansatz is introduced and used to fit the results from simulations covering the range from fully stochastic to the zero-noise limit. [S1063-651X(98)05909-1]

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I. INTRODUCTION

The Eden model [1], which was originally introduced to model the growth of cell colonies, is one of the simplest and most widely studied [1–24] growth models and has become a paradigm for studying self-affine fractal geometry and scaling in the growth of rough surfaces [25–27]. In the simplest variant of the model (Eden A) aggregate particles are added one at a time to randomly selected sites on the surface of a growing cluster [3,4]. The aggregate sites and surface sites are situated at the vertices of a regular lattice in on-lattice simulations. The initial cluster is usually taken to be a single aggregate site in the plane (radial growth) or a line of aggregate sites in the half plane (substrate growth). Most studies of the Eden model have been concerned with describing the asymptotic properties of the surface of the cluster. One of the schemes that has been introduced to better reveal these properties is noise reduction. Noise reduction is implemented by associating a counter (initially set to zero) with each of the surface sites and incrementing the counter by one each time the associated surface site is selected for growth [19–21,23]. An aggregate particle is then added at a surface site when its associated counter reaches a prescribed value m. Increasing values of m lead to increasingly smoother interfaces [22,24]. It is widely believed that noise reduction reveals asymptotic surface properties at smaller system sizes without affecting the surface scaling exponents. This tenant is investigated in this paper for the Eden A model with growth from a seed on a square lattice.

II. SURFACE SCALING ANSATZ

Extensive studies of Eden growth from a substrate [6,9,10,12,14–17,19–21,23] have identified scaling exponents α and z relating the surface thickness w to the substrate width L and the mean surface height $\langle h\rangle$. For a cluster with N aggregate particles and $\mathcal{N}$ surface sites this relationship has the form [8]

$$w \sim L^{\alpha} f\left(\frac{\langle h\rangle}{L^{z}}\right), \tag{2.1}$$

where the width is defined by

$$w^2=\langle h^2\rangle-\langle h\rangle^2=\frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}} h_i^2-\left(\frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}} h_i\right)^2 \tag{2.2}$$

and the surface function $f(x)$ has the properties

$$f(x)\propto\begin{cases} x^{\alpha/z} & \text{for } x\ll 1\\ \text{const} & \text{for } x\gg 1.\end{cases} \tag{2.3}$$

It follows from the scaling laws, Eqs. (2.1), (2.3), that for L large the surface of Eden clusters on a two-dimensional substrate is a self-affine fractal with Hurst exponent α and fractal dimension $2-\alpha$.

Collective evidence from numerical simulations [6,9,11,12,15,16] and algebraic calculations [13] suggest the values $\alpha\simeq 1/2$ and $z\simeq 3/2$ (or $\beta\equiv\alpha/z\simeq 1/3$) for two-dimensional Eden growth on a substrate. However, it should be pointed out that the numerical results are not definitive due to finite size effects and the algebraic results may not be entirely applicable since they are based on a continuum model that includes surface relaxations.

For an Eden cluster growing in a circular geometry with N aggregate particles, $\mathcal{N}$ surface sites, and an average radius $\langle R\rangle$ (which grows linearly with time) the interface width

$$w^2=\langle R^2\rangle-\langle R\rangle^2=\frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}} R_i^2-\left(\frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}} R_i\right)^2 \tag{2.4}$$

is expected to scale as [13]

$$w\sim\langle R\rangle^{\beta}. \tag{2.5}$$

Numerical simulations of Eden A clusters on the square lattice with up to $N\approx 10^5$ aggregate particles again suggest $\beta\simeq 1/3$. However, very large simulations for $N\approx 10^7$ [12] and $N\approx 10^9$ [16] reveal the increasingly dominant effects of lattice anisotropy where eventually it is expected [16] that $w\sim\langle R\rangle$. In this paper we have carried out brute force simulations of the Eden A model on the square lattice starting from a single aggregate site over a range of m from the fully stochastic limit $m=1$ to the zero-noise limit $m\rightarrow\infty$ [24]. The brute force calculations avoid the possibility of numerical bias from, e.g., quadrant boundary effects [12] or multiply selected surface sites [16]. The results of the simulations are shown to be consistent with a two-exponent scaling ansatz of the form

$$w(N,m)\sim\begin{cases} a(m)N^{1/6} & \text{for } N\ll N^*(m)\\ b(m)N^{1/2} & \text{for } N\gg N^*(m),\end{cases} \tag{2.6}$$

where $N^*(m)$ denotes an empirical crossover number of aggregate particles for a given m. This is in agreement with the expectation that noise reduction does not affect the values of the scaling exponents; however, the crossover value

$$N^*=\left(\frac{a(m)}{b(m)}\right)^3 \tag{2.7}$$

is m dependent. The two-exponent scaling ansatz can also be written in the functional form

$$w(N,m)\sim b(m)N^{1/2}g\left(\frac{N}{N^*(m)}\right), \tag{2.8}$$

where

$$g(x)=\begin{cases} x^{-1/3} & \text{for } x\ll 1\\ 1 & \text{for } x\gg 1.\end{cases} \tag{2.9}$$

III. ZERO-NOISE LIMIT

In the zero-noise limit $m\to\infty$, Eden A clusters on a square lattice grow in layers as a compact diamond with [24]

$$N_k=2k^2-2k+1,\quad k=1,2,\ldots \tag{3.1}$$

aggregate particles. In this limit there is no stochastic growth so that

$$\lim_{m\to\infty} N^*(m)\to 0 \tag{3.2}$$

and

$$\lim_{m\to\infty} w(N,m)\sim b(\infty)N^{1/2}. \tag{3.3}$$

A simple approximation to $b(\infty)$ can be found from the continuum expression for a diamond in polar coordinates:

$$R(\theta)=\frac{\sqrt{N}}{\sqrt{2}[|\sin(\theta)|+|\cos(\theta)|]}. \tag{3.4}$$

The averages over θ can be calculated exactly yielding

$$w\sim\sqrt{\frac{1}{\pi}-\frac{4(\tanh^{-1}(1/\sqrt{2}))^2}{\pi^2}}N^{1/2} \tag{3.5}$$

and hence $b(\infty)\approx 0.05896,\ldots$.

IV. NUMERICAL RESULTS

The results described in this section summarize data from our numerical simulations of ensembles of Eden A clusters starting from a single seed on the square lattice. Each ensemble consists of 100 Monte Carlo simulations of the Eden A model for a fixed value of the noise-reduction parameter m. The surface width, Eq. (2.4), is averaged over the ensemble copies to obtain the surface width as a function of N for a given m.

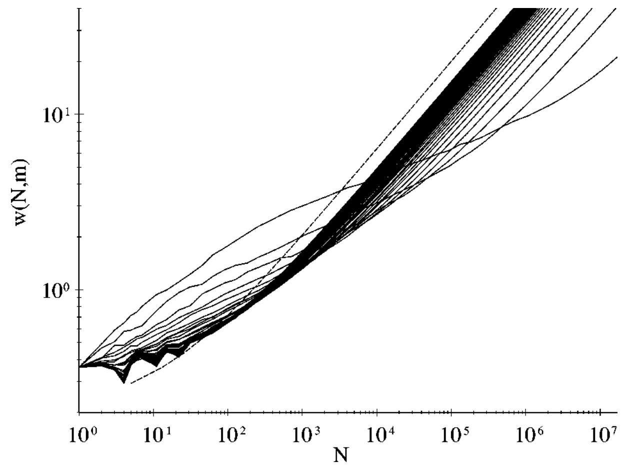

FIG. 1. Plots of the surface width vs the number of aggregate particles for different values of the noise reduction parameter at one unit intervals in the range $m\in[1,64]$ and for the zero-noise limit (dashed curve). Note the logarithmic scale on each of the axes.

Figure 1 shows plots of the ensemble averaged surface width versus the number of aggregate particles (using a log-log scale) for m at one unit intervals in the range $m\in[1,64]$ and for $m\to\infty$ (dashed line). The curves for increasing values of m are from right to left on the right-hand side of the plot and upper to lower on the left-hand side of the plot. The peaks in the sawtooth pattern for large m and small N occur at exact ''diamond numbers,'' Eq. (3.1). The surface width data were fit to the two-exponent scaling ansatz, Eq. (2.6). In Fig. 2 the best fit estimates for (a) $a(m)$ and (b) $b(m)$ are plotted against m at one unit intervals in the range $m\in[1,64]$. Figure 2(b) also shows (dashed line) the asymptotic value of $b(\infty)$ obtained from the calculation in the zero-noise limit, Eq. (3.5). The surface profile of very large Eden clusters at $m=1$ is slightly anisotropic and well fit (in the first quadrant) by

$$R(\theta)=\langle R\rangle+A\cos(4\theta), \tag{4.1}$$

where $\langle R\rangle\approx\sqrt{N/\pi}$ is the average radius of the cluster and A is the amplitude of the anisotropy (about 1% of $\langle R\rangle$ [16]). The anisotropic profile, Eq. (4.1), has a surface width

$$w\sim\frac{A}{\sqrt{2\pi}}N^{1/2}. \tag{4.2}$$

Our value $b(1)\approx 0.005$ is thus consistent with a slight anisotropy of the order of $A\approx\pm 1\%\langle R\rangle$.

The functional form of the two-exponent scaling ansatz, Eq. (2.8), is clearly revealed in Fig. 3 where we have collapsed the surface width data for all N and m values onto a single curve by plotting

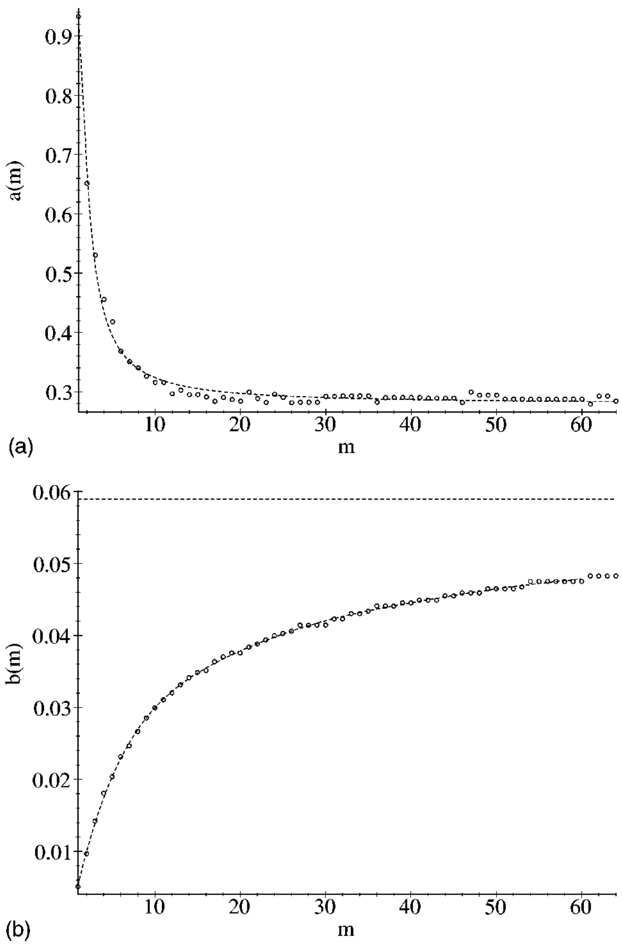

FIG. 2. Plots of the surface width scaling coefficients in Eq. (2.6) as a function of the noise reduction parameter: (a) $a(m)$ versus m and (b) $b(m)$ versus m.

$$\frac{w(N,m)}{b(m)N^{1/2}} \quad \text{versus} \quad \frac{N}{N^*(m)}.$$

The dashed line in Fig. 3 has a slope $\gamma = -1/3$.

V. DISCUSSION

In this paper we showed that the surface width of noise-reduced Eden A clusters grown from a seed on a square lattice scales with the number of aggregate particles according to a two-exponent relation. The exponents $\nu = 1/6$ for $N < N^*$ and $\nu = 1/2$ for $N > N^*$ were found to be independent of the noise reduction parameter m but the crossover value N^* was found to decrease monotonically with m. These results support the tenants that (i) noise reduction does not affect the scaling exponents in Eden-like growth models and (ii) increasing noise reduction decreases the size of clusters needed for observing the limiting large N scaling behavior. In particular, provided the scaling coefficients do not vanish in the limit $m \to \infty$, the large N scaling exponents could be found rather simply from exact (algebraic or numerical) calculations in the zero-noise limit. On the other hand, intermediate scaling results from finite m simulations would have to be interpreted with some caution particularly in cases where the growth is characterized by multiexponent scaling laws and multiple m-dependent crossover values. It is anticipated that a similar two-exponent scaling relation to Eq. (2.8) with the same two scaling exponents but different coefficients $a(m)$ and $b(m)$ could be used to characterize on-lattice (e.g., square, triangular, or honeycomb) growth of other variants of the Eden model (e.g., Eden B and Eden C).

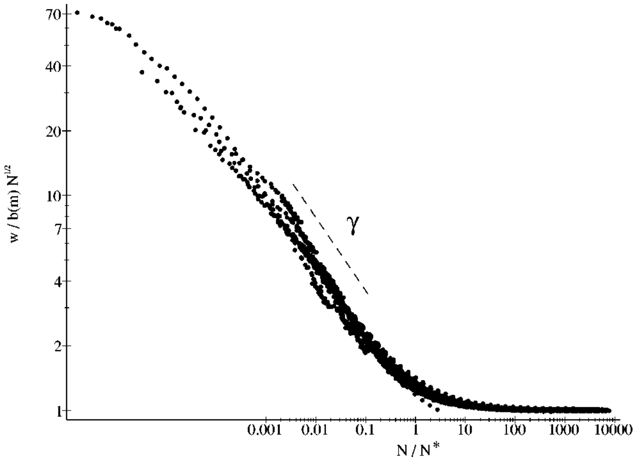

FIG. 3. Plots of $w(N,m)/b(m)N^{1/2}$ vs $N/N^*(m)$ for each integer value of m in the range $[1,64]$. The surface width $w(N,m)$ is averaged over 100 different Monte Carlo simulations at that value of N and m. Note the logarithmic scale on each of the axes.

ACKNOWLEDGMENT

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